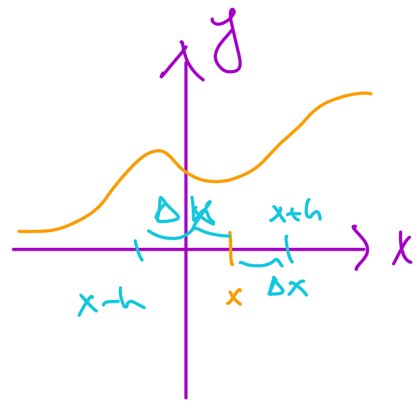
§2 "Focus on theory"

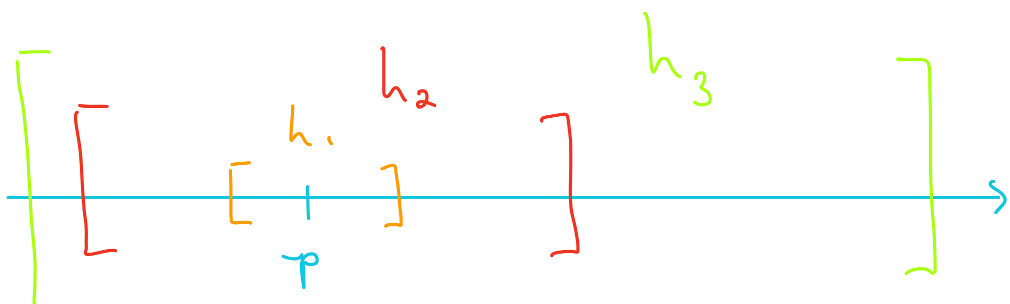
Rmk: For any function f, we define the "derivative function", f', by



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. We say f is "differentiable at p" if $f'(p) \in (-\infty, \infty)$.

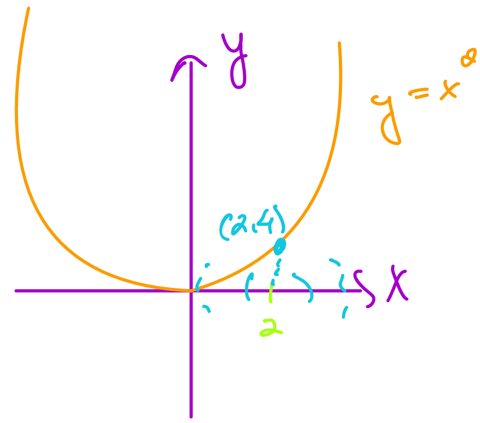
Rmk: "$\lim_{h \to 0}$" describes the process of taking intervals of length h as h gets closer to zero.



$$0 < \cdots < h_1 < h_2 < h_3 < \cdots$$

Ex: Investigate $\lim\limits_{x\to 2} x^2$.

$$\lim_{x\to 2} x^2 = 2^2 = 4$$



Ex: Use algebra to find $\lim\limits_{h\to 0} \dfrac{(3+h)^2-9}{h}$.

Sol:

$$\lim_{h\to 0}\frac{(3+h)^2-9}{h} = \lim_{h\to 0}\frac{\cancel{9}+6h+h^2-\cancel{9}}{h}$$

$$= \lim_{h\to 0}\frac{6h+h^2}{h} = \lim_{h\to 0}\frac{\cancel{h}(6+h)}{\cancel{h}}$$

$$= \lim_{h\to 0}(6+h) = 6+0 = 6$$

$h_1 > h_2 > h_2 > \cdots > 0 \Rightarrow 6+h_1 > 6+h_2 > \cdots \Rightarrow 6+h \xrightarrow{h\to 0} 6$

Ex: $\lim\limits_{x\to 1} x^2+1 =$

$x_1 > \cdots > x_2 > \cdots > 1$

$1 > \cdots > x_2 > x_1$

$\Rightarrow x_1^2+1, \cdots$

$1^2+1=2$

Ex: $\lim_{x \to 0} \frac{1}{x}$, $\lim_{x \to 1} \frac{1}{x^2-1}$

[$p \in dom(f) \Rightarrow \lim_{x \to p} f(x) = f(p)$

$p \notin dom(f) \Rightarrow \lim_{x \to p} f(x) \neq f(p)$ ← not a value

Ex: Show the derivative of $f(x) = x^2$ is $f'(x) = 2x$.

Sol: $f'(x) \overset{(def.)}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h = 2x + 0 = 2x$$

Ex: Find $f'(x)$ where $f(x) = 3x^2 - 2$.

Sol: $f'(x) = \lim_{h \to 0} \frac{\boxed{f(x+h)} - \boxed{f(x)}}{h}$ (def of $f'(x)$)

$$= \lim_{h\to 0} \frac{\boxed{3(x+h)^2 - 2} - \boxed{(3x^2-2)}}{h}$$

$$= \lim_{h\to 0} \frac{3(x^2+2xh+h^2) - 2 - 3x^2 + 2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-2} - \cancel{3x^2} \cancel{+2}}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(6x+3h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 6x + 3h$$

$$= 6x + \cancel{3\cdot 0}^{\,0}$$

$$= 6x$$

Problems: 19 to 22, 34 to 43.

(a1) $\lim_{h\to 0} \frac{(h+1)^2 - 1}{h}$

(42) Show the following.

If $f(x) = \frac{1}{x}$, then

$f'(x) = -\frac{1}{x^2}$.

(Hint: Use def of $f'(x)$, see above)

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$