§3.1: Derivatives for polynomials

Obj: We derive rules that tell us how to compute the derivative of polynomials.

Disc.: We will go thru "proofs" for the rules only to help justify "where they came from".

Warm up: Compute the derivative of $f(x) = 2x^2 + x$.

(Hint: $\lim\limits_{h \to 0}$)

$$\lim_{h \to 0} \frac{2(x+h)^2 + (x+h) - (2x^2 + x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{x} + h - \cancel{2x^2} - \cancel{x}}{h}$$

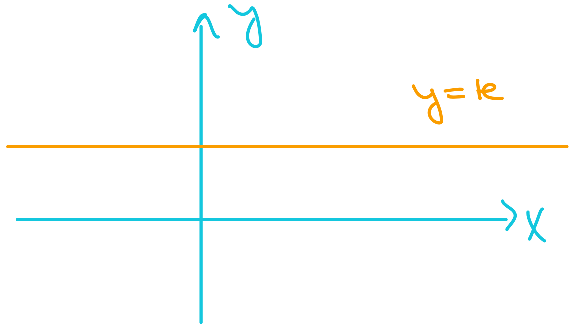
$$= \lim_{h \to 0} \frac{4xh + 2h^2 + h}{h} = \lim_{h \to 0} 4x + 2h + 1 = 4x + 1.$$

Thm: (Constant) Let $k \in (-\infty, \infty)$. If $f(x) = k$, then $f'(x) = 0$.

Pf: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k - k}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0. \quad \blacksquare$$

Rmk:



← has slope zero at all tangent lines

Ex:

$f(x)=5 \implies f'(x)=0$

$h(x)=1001 \implies h'(x)=0$

$g(x)=\pi \implies g'(x)=0.$

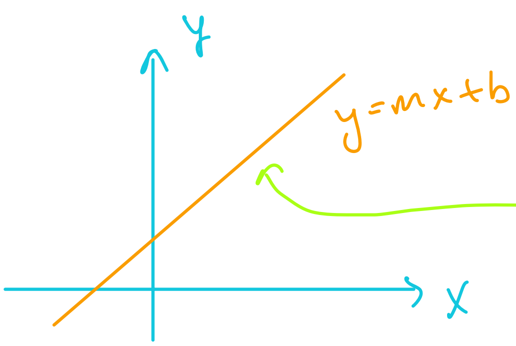
Lem: (Linear) Let $b,m \in (-\infty,\infty)$ where $m \neq 0$. If $f(x)=mx+b$, then $f'(x)=m$.

Pf: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(m(x+h)+b)-(mx+b)}{h}$

$=\lim_{h\to 0}\frac{\cancel{mx}+mh+\cancel{b}-\cancel{mx}-\cancel{b}}{h}=\lim_{h\to 0}\frac{m\cancel{h}}{\cancel{h}}$

$=\lim_{h\to 0} m = m.$ ∎

Rmk:



has slope m at all tangent lines

Ex: $f(x) = 2x+3 \Longrightarrow f'(x) = 2$

$g(x) = 50x+1 \Rightarrow g'(x) = 50$

$h(x) = 4x \Longrightarrow h'(x) = 4$
$= 4x+0$

Rmk: Recall we have different notation for derivatives: $f'(x)$ or $\frac{d}{dx} f(x)$. If we write the two results above in this way, then

- $\frac{d}{dx} f = 0$ if $f(x) = k$ for $k \in (-\infty, \infty)$

- $\frac{d}{dx} f = m$ if $f(x) = mx+b$ a linear function

Thm: (Quadratics) Let $a, b, c \in (-\infty, \infty)$ where $a \neq 0$. If $f(x) = ax^2+bx+c$, then $\frac{d}{dx} f(x) = 2ax+b$.

Pf: (See Exam 1)

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a(x+h)^2+b(x+h)+c-(ax^2+bx+c)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{ax^2}+2axh+ah^2+\cancel{bx}+bh+\cancel{c}-\cancel{ax^2}-\cancel{bx}-\cancel{c}}{h}$$

$$= \lim_{h\to 0} \frac{2axh+ah^2+bh}{h} = \lim_{h\to 0} 2ax+ah+b = 2ax+b. \blacksquare$$

Ex: $f(x) = 2x^2 + 1 \Rightarrow 4x$

$\left[\frac{d}{dx} f = 2ax + b \right]$

$g(x) = 3x^2 + 4x + 2 \Rightarrow 2 \cdot 3x + 4 = 6x + 4$

$h(x) = 3x^2 + 4x + 1 \Rightarrow 6x + 4$

Thm: (Const. multiple) Let $c \in (-\infty, \infty)$. If $f(x)$ is a function, then

$$\frac{d}{dx}\left(c \cdot f(x)\right) = c \cdot \frac{d}{dx} f(x).$$

$\left[\frac{d}{dx} : \{\text{Functions}\} \to \{\text{Functions}\} \right]$

Pf: $$\frac{d}{dx}\left(cf(x)\right) \overset{(\text{def})}{=} \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \to 0} \frac{c\left(f(x+h) - f(x)\right)}{h}$$

$$= c \cdot \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} = c \cdot \frac{d}{dx} f(x)$$

Ex: $f(x) = 2x^2 + 2x + 2 = 2(x^2 + x + 1)$

$$\frac{df}{dx} = \frac{d}{dx}\left(2 \cdot (x^2 + x + 1)\right) = 2 \cdot \frac{d}{dx}(x^2 + x + 1) = 2(2x + 1) = 4x + 2$$

$g(x) = 12x^2 + 4x + 4 = 4(3x^2 + x + 1)$

$$\frac{dg}{dx} = \frac{d}{dx}\left(4(3x^2 + x + 1)\right) = 4 \frac{d}{dx}\left(3x^2 + x + 1\right) = 4 \cdot (6x + 1) = 24x + 4$$

$h(x) = 21x^2 + 12x + 3 = 3(7x^2 + 4x + 1)$

$$\frac{dh}{dx} = \frac{d}{dx}\left(3 \cdot (7x^2 + 4x + 1)\right) = 3 \cdot \frac{d}{dx}(7x^2 + 4x + 1) = 3 \cdot (14x + 4) = 42x + 12$$

Thm: (Sums and differences) If $f(x)$ and $g(x)$ are functions, then $\lceil a(b+c) = ab + ac \rfloor$

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x).$$

Pf: We work the case for "+" as "−" is similar.

$$\frac{d}{dx}\left(f(x)+g(x)\right) = \lim_{h\to 0} \frac{\left(f(x+h)+g(x+h)\right) - \left(f(x)+g(x)\right)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x) + g(x+h)-g(x)}{h}$$

$\lceil \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \rfloor$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$

$$= \underbrace{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}} + \underbrace{\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}$$

$\left(\lim (f+g) = \lim f + \lim g\right)$

$$= \frac{df}{dx} + \frac{dg}{dx}. \blacksquare$$

Ex: $f(x) = x^2+2x+3$, $g(x) = 3x+1$

$$\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}\left(x^2+2x+3+3x+1\right)$$

$$= \frac{d}{dx}\left(x^2+5x+4\right) = 2x+5$$

$$\frac{d}{dx} f(x) + \frac{d}{dx} g(x) = \frac{d}{dx}\left(x^2+2x+3\right) + \frac{d}{dx}\left(3x+1\right)$$

$= 2x + 2 + 3 = 2x + 5.$

Ex: $\frac{d}{dx}(x^2 + 2x + 1 + 5x^2 + 3x + 7) =$

$\frac{d}{dx}(x^2 + 2x + 1) + \frac{d}{dx}(5x^2 + 3x + 7) =$

$2x + 2 + 10x + 3 = 12x + 5.$

Lem: (Powers) Let $n \in (-\infty, \infty)$. If $f(x) = x^n$, then

$\frac{d}{dx} f(x) = n x^{n-1}$.

Pf: We might come back to this later. The proof is "trickier". ☒

$\left[\frac{d}{dx}(a x^n) = a \cdot n \cdot x^{n-1} \right]$

Ex: $f(x) = x^2 \Rightarrow \frac{df}{dx}(x) = 2x$

$g(x) = 2x^3 \Rightarrow \frac{d}{dx} g(x) = \frac{d}{dx}(2x^3) = 2\frac{d}{dx}(x^3) = 2.3x^2 = 6x^2$

$h(x) = x^{\frac{1}{3}} \Rightarrow \frac{d}{dx} h(x) = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3} x^{-\frac{2}{3}}$

$i(x) = 3x^{\frac{1}{3}} \Rightarrow \frac{d}{dx} i(x) = 3 \cdot \frac{d}{dx} x^{\frac{1}{3}} = \cancel{3} \cdot \left(\frac{1}{\cancel{3}} x^{-2/3}\right) = x^{-2/3}$

$j(x) = \sqrt{x} \Rightarrow \frac{d}{dx} j(x) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Lem: (Polynomials) If n is a positive integer, then

$$\frac{d}{dx}(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) =$$

$$a_n \cdot n x^{n-1} + a_{n-1} \cdot (n-1) x^{n-2} + \cdots + a_2 x + a_1.$$

Pf: $\frac{d}{dx}(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) =$

$\frac{d}{dx}(a_n x^n) + \frac{d}{dx}(a_{n-1} x^{n-1}) + \cdots + \frac{d}{dx}(a_1 x) + \cancel{\frac{d}{dx}(a_0)}^{0} =$

$a_n \frac{d}{dx} x^n + a_{n-1} \frac{d}{dx} x^{n-1} + \cdots + a_1 \frac{d}{dx} x^0 =$

$a_n \cdot n \cdot x^{n-1} + a_{n-1} \cdot (n-1) \cdot x^{n-2} + \cdots + a_1 \qquad (x^0 = 1)$. $\boxtimes$

Ex: $f(x) = x^3 + x + 1 \Rightarrow \frac{d}{dx}(x^3 + x + 1) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x) + \cancel{\frac{d}{dx}(1)}^{0}$

$= 3x^2 + 1$

$g(x) = 3x^4 + 2x + 2 \Rightarrow \frac{d}{dx}(3x^4 + 2x + 2) = \frac{d}{dx}(3x^4) + \frac{d}{dx}(2x) + \cancel{\frac{d}{dx}(2)}^{0}$

$= 3\frac{d}{dx}(x^4) + 2\frac{d}{dx}(x)$

$= 12x^3 + 2$

$h(x) = (x-1)(x+1) \Rightarrow \frac{d}{dx}\big((x-1)(x+1)\big)$

$= \frac{d}{dx}(x^2 - 1) = \frac{d}{dx}(x^2) + \cancel{\frac{d}{dx}(-1)}^{0}$

$= 2x$.

Summary: $n, k, m, b \in (-\infty, \infty)$ where $m \neq 0$

- $\frac{d}{dx}(k) = 0$
- $\frac{d}{dx}(x^n) = n x^{n-1}$
- $\frac{d}{dx}(mx + b) = m$
- $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

* Work through problems in §3.1 *

Problems 1 to 38 → Choose 12 problems.

Groups of four. Turn in one assignment w/ all names on it.