§3.2: Exponential & logarithmic functions

Obj: We will study how to compute the derivative of an exponential and logarithmic function.

Warm up: Compute $\frac{d}{dx}\left(\sqrt{\frac{1}{x^3}}\right)$ and $\frac{d}{d\theta}\left(\frac{1}{\sqrt[3]{\theta}}\right)$.

Sol:

$$\frac{d}{dx}\left(\sqrt{\frac{1}{x^3}}\right) = \frac{d}{dx}\left(\left(\frac{1}{x^3}\right)^{\frac{1}{2}}\right) = \frac{d}{dx}\left(x^{-\frac{3}{2}}\right) = -\frac{3}{2}x^{-\frac{3}{2}-1} = -\frac{3}{2}x^{-\frac{5}{2}}$$

$$\frac{d}{d\theta}\left(\frac{1}{\sqrt[3]{\theta}}\right) = \frac{d}{d\theta}\left(\frac{1}{\theta^{\frac{1}{3}}}\right) = \frac{d}{d\theta}\left(\theta^{-\frac{1}{3}}\right) = -\frac{1}{3}\theta^{-\frac{1}{3}-1} = -\frac{1}{3}\theta^{-\frac{4}{3}}$$

Recall: Let $n, k \in (-\infty, \infty)$ and f, g be functions.

- $\frac{d}{dx}(k) = 0$
- $\frac{d}{dx}(x^n) = n x^{n-1}$
- $\frac{d}{dx}(f \pm g) = \frac{d}{dx}f \pm \frac{d}{dx}g$
- $\frac{d}{dx}(k \cdot f) = k \frac{d}{dx}f$

Thm: $\frac{d}{dx}(a^x) = (\ln(a)) \cdot a^x$

Pf: (Just for intuition)

$$\frac{d}{dx}(a^x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h - 1)}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x \cdot \ln(a). \blacksquare$$

This step needs care to show, and one can use $a^x = e^{x\ln(a)}$

Ex: Compute $\frac{d}{dx}(3^x)$ and $\frac{d}{dx}(3^x + 2)$.

Sol: $\frac{d}{dx}(3^x) = \ln(3) \cdot 3^x$ (exp. rule)

$\frac{d}{dx}(3^x + 2) = \frac{d}{dx}(3^x) + \overset{0}{\cancel{\frac{d}{dx}(2)}}$ ($\frac{d}{dx}$ of sum)

$= \ln(3) \cdot 3^x$ ($\frac{d}{dx}$ of exp and const)

Ex: Compute $\frac{d}{dx}(3x^2 + 3^x + 5)$.

Sol: $\frac{d}{dx}(3x^2 + 3^x + 5) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(3^x) + \overset{0}{\cancel{\frac{d}{dx}(5)}}$

$= 6x + \ln(3) \cdot 3^x$

Ex: Find the derivative of the following function:

$$f(x) = \frac{2}{x^3} + x^3 + 3^x + 2.$$

Sol:

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{2}{x^3} + x^3 + 3^x + 2\right)$$

$$= \frac{d}{dx}\left(\frac{2}{x^3}\right) + \frac{d}{dx}(x^3) + \frac{d}{dx}(3^x) + \cancel{\frac{d}{dx}(2)}^{\,0}$$

$$\frac{d}{dx}(2x^{-3}) = 2\frac{d}{dx}(x^{-3}) = 2\cdot(-3)\,x^{-3-1}$$

$$= \frac{d}{dx}(2\cdot x^{-3}) + 3x^2 + \ln(3)\cdot 3^x$$

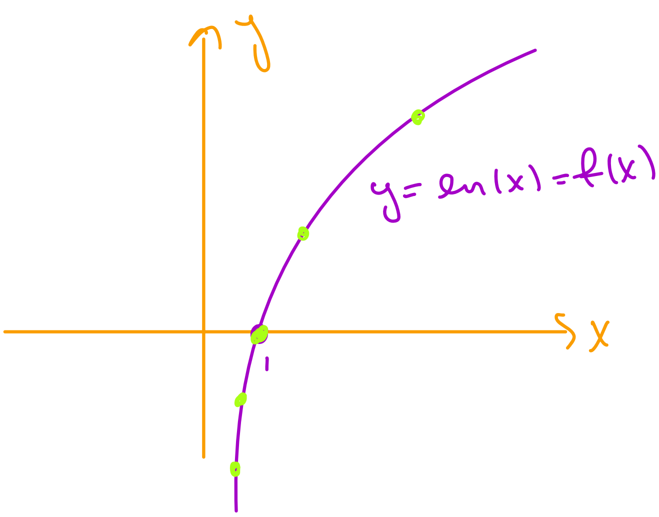
$$= 2\cdot(-3)\,x^{-4} + 3x^2 + \ln(3)\,3^x$$

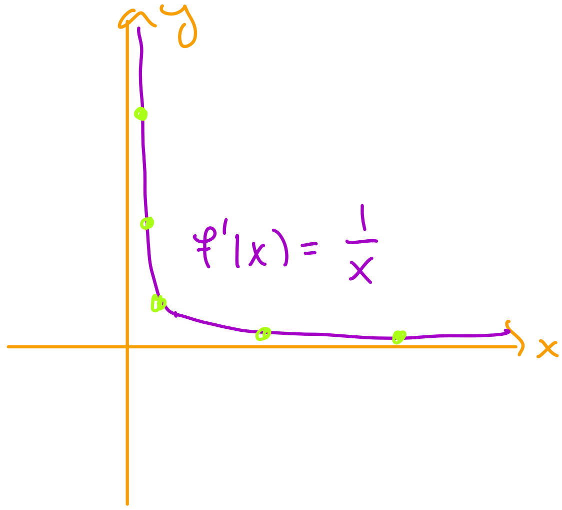
$$= -6x^{-4} + 3x^2 + \ln(3)\cdot 3^x$$

Lem: $\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}$.

Pf: We *might* see this later. A reference is the "Forum on theory" section in Chp 3. ☒

Rmk:





Ex: Differentiate $g(x) = 5\ln x + 7e^x - 4x^2 + 5$.

Sol: $\frac{d}{dx} g(x) = \frac{d}{dx}(5\ln x + 7e^x - 4x^2 + 5)$

$= \frac{d}{dx}(5\ln x) + \frac{d}{dx}(7e^x) + \frac{d}{dx}(-4x^2) + \cancel{\frac{d}{dx}(5)}^{0}$

$= 5\frac{d}{dx}(\ln(x)) + 7\frac{d}{dx}(e^x) - 4\frac{d}{dx}(x^2)$

$= 5 \cdot \frac{1}{x} + 7 \cdot e^x - 4 \cdot 2 \cdot x$

$= \frac{5}{x} + 7e^x - 8x$

$\left[\frac{d}{dx}(e^x) = \ln(e) \cdot e^x = e^x \right]$

Summary:

- if $a \in (0, \infty)$, then $\frac{d}{dx}(a^x) = \ln(a)\, a^x$
- $\frac{d}{dx}(e^x) = e^x$ (e in Euler's constant)
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$

* §3.2: 1 to 34, do 15