§3.4: Product and quotient rule

obj: We compute the derivative of a product and quotient of two functions.

Warm up: Compute

$$\frac{d}{dx}\left((5x-6)^3\right) = \frac{d}{dx}(g(x))\Big|_{x=f(x)} \cdot \frac{d}{dx} f(x)$$

inside $= f(x) = 5x-6$

outside $= g(x) = x^3$

$$= 3x^2\Big|_{x=f(x)} \cdot 5$$

$$= 3\, f(x)^2 \cdot 5$$

$$= 15\,(5x-6)^2$$

$$\left[\frac{d}{dx}\left((5x-6)^3\right)\right] = 3(5x-6)^2 \cdot \frac{d}{dx}(5x-6) = 15\,(5x-6)^2$$

✓

Thm: (Product) If $f(x)$ and $g(x)$ are two functions, then

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = \frac{d}{dx}(f)\cdot g + f\cdot\frac{d}{dx}g$$

or equivalently,

$$(f\cdot g)' = f'\cdot g + f\cdot g'.$$

Ex: $$\frac{d}{dx}\left(x^2 e^{2x}\right) = \frac{d}{dx}(x^2)\cdot e^{2x} + x^2\, \boxed{\frac{d}{dx}\left(e^{2x}\right)}$$

chain rule

$$= 2x\cdot e^{2x} + x^2\cdot \boxed{e^{2x}\cdot\frac{d}{dx}(2x)}$$

$$= 2x e^{2x} + x^2 e^{2x}\cdot 2$$

$$= 2xe^{2x}(1+x)$$

Ex: $\frac{d}{dt}\left(t^3\ln(t+1)\right) = \frac{d}{dt}(t^3)\ln(t+1) + t^3\cdot\boxed{\frac{d}{dt}\ln(t+1)}$

$$= 3t^2\ln(t+1) + t^3\cdot\boxed{\frac{1}{t+1}\cdot\cancelto{1}{\frac{d}{dt}(t+1)}}$$

chain rule

$$= 3t^2\ln(t+1) + \frac{t^3}{t+1}$$

$$= t^2\left(3\ln(t+1) + \frac{t}{t+1}\right)$$

Ex: $\frac{d}{dx}\left((3x^2+5x)e^x\right) =$

$$\frac{d}{dx}(3x^2+5x)e^x + (3x^2+5x)\cdot\frac{d}{dx}(e^x) =$$

$$\left(\frac{d}{dx}(3x^2) + \frac{d}{dx}(5x)\right)e^x + (3x^2+5x)e^x =$$

$\ulcorner \frac{d}{dx}(a^x) = \ln(a)\cdot a^x \lrcorner$

$+ \ln(e) = 1 \lrcorner$

$$(6x+5)e^x + (3x^2+5x)e^x =$$

$$(11x+3x^2+5)e^x$$

Thm: (Quotient) If $f(x)$ and $g(x)$ are two functions, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f)\cdot g - f\cdot\frac{d}{dx}(g)}{g^2}$$

"(dee high low − high dee low) / square below"

or equivalently,

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Ex:

$$\frac{d}{dx}\left(\frac{5x^2}{x^3+1}\right)=\frac{\frac{d}{dx}(f)\cdot g-f\cdot\frac{d}{dx}(g)}{g^2}$$

$$=\frac{\frac{d}{dx}(5x^2)\cdot(x^3+1)-5x^2\frac{d}{dx}(x^3+1)}{(x^3+1)^2}$$

$$=\frac{10x(x^3+1)-5x^2\cdot 3x^2}{(x^3+1)^2}=\frac{10x^4+10x-15x^4}{(x^3+1)^2}$$

$$=\frac{-5x^4+10x}{(x^2+1)^2}$$

Ex:

$$\frac{d}{dt}\left(\frac{1}{1+e^t}\right)=\frac{\cancel{\overset{0}{\frac{d}{dt}(1)}\cdot(1+e^t)}-1\cdot\frac{d}{dt}(1+e^t)}{(1+e^t)^2}$$

$$=-\frac{\frac{d}{dt}(1+e^t)}{(1+e^t)^2}=\frac{-e^t}{(1+e^t)^2}$$

[$f(x)$ function , c constant

$$\frac{d}{dx}(f(x)+c)=\frac{d}{dx}(f(x))+\overset{0}{\cancel{\frac{d}{dx}(c)}}=\frac{d}{dx}f(x)$$]

Ex:

$$\frac{d}{dx}\left(\frac{e^x}{x^2}\right)=\frac{\frac{d}{dx}(e^x)\,x^2-e^x\frac{d}{dx}(x^2)}{x^4}$$

$$=\frac{e^x\cdot x^2-e^x\cdot 2x}{x^4}$$

* We used quotient rule *

$$=\frac{e^x\cancel{x}(x-2)}{x^{\cancel{4}}}=\frac{e^x(x-2)}{x^3}$$

$$=\frac{e^x\cdot x}{x^3}-\frac{2e^x}{x^3}=\frac{e^x}{x^2}-\frac{2e^x}{x^3}$$

Alt: $\frac{d}{dx}\left(\frac{e^x}{x^2}\right) = \frac{d}{dx}\left(e^x \cdot x^{-2}\right)$

* product rule can be used *

$= \frac{d}{dx}(e^x) \cdot x^{-2} + e^x \frac{d}{dx}(x^{-2})$

$= e^x \cdot x^{-2} + e^x(-2x^{-3})$

$= \frac{e^x}{x^2} - \frac{2e^x}{x^3}$

Rmk: $f(x), g(x)$ are two functions and nonzero

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x) \cdot \frac{1}{g(x)}\right)$$

quotient rule

$$= \frac{d}{dx}\left(f(x) \cdot g(x)^{-1}\right)$$

product rule

Q: Compute $\frac{d}{dt}\left(\frac{1-t}{1+t}\right)$.

$$\frac{d}{dt}\left(\frac{1-t}{1+t}\right) = \frac{\overset{-1}{\cancel{\frac{d}{dt}(1-t)}}(1+t) - (1-t)\overset{1}{\cancel{\frac{d}{dt}(1+t)}}}{(1+t)^2}$$

$$= \frac{-1(1+t)-(1-t)}{(1+t)^2} = \frac{-1-\cancel{t}-1+\cancel{t}}{(1+t)^2}$$

$$= -\frac{2}{(1+t)^2}$$

<u>Alt</u>: Using product rule

$$\frac{d}{dt}\left(\frac{1-t}{1+t}\right) = \frac{d}{dt}\left((1-t)(1+t)^{-1}\right)$$

$$= \frac{d}{dt}\left(\overset{-1}{\cancel{(1-t)}}\right)\cdot(1+t)^{-1} + (1-t)\,\boxed{\frac{d}{dt}\left((1+t)^{-1}\right)}$$

chain rule

$$= -(1+t)^{-1} + (1-t)\cdot\boxed{\left(-(1+t)^{-2}\cdot 1\right)}$$

$$= \boxed{-\frac{1}{1+t}} + \frac{-(1-t)}{(1+t)^2}$$

$$= \boxed{-\frac{(1+t)}{(1+t)^2}} + \frac{-1+t}{(1+t)^2}$$

$$= \frac{-1 - \cancel{t} - 1 + \cancel{t}}{(1+t)^2} = \frac{-2}{(1+t)^2}$$

Assignment to turn in: Compute the following.

1. $\frac{d}{dt}\left(t^2 \ln(t)\right)$

2. $\frac{d}{dt}\left(\frac{t}{e^t}\right)$

3. $\frac{d}{dt}\left(e^{\sqrt{t}}\right)$